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LETTER TO THE EDITOR

Chiral Potts model on a Cayley tree with complete and incomplete devil's staircase

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Abstract. The three-state chiral Potts model on a Cayley tree is analysed in the limit of infinitely large coordination number. The fractal dimensionalities of the wavenumber against chiral field curves are computed. It is shown that they change from a complete to an incomplete devil's staircase as the temperature is raised. Commensurate phase boundaries are determined analytically for high and low temperatures, and the discommensurations are shown to result from tangent bifurcations.

Systems exhibiting spatially modulated structures, commensurate or incommensurate with the underlying lattice, are of current interest in condensed matter physics (see e.g. Bak 1982). Among the idealised systems for modulated ordering, the axial next-nearest-neighbour Ising (ANNNI) model, originally introduced by Elliott (1961) to describe the sinusoidal magnetic structure of Erbium, and the chiral Potts model, introduced by Ostlund (1981) and Huse (1981) in connection with monolayers adsorbed on rectangular substrates, have been studied extensively by a variety of techniques. A particularly interesting and powerful method is the study of modulated phases through the measure-preserving map generated by the mean-field equations, as applied by Bak (1981) and Jensen and Bak (1983) to the ANNNI model. The main drawback of the method lies in the fact that thermodynamic solutions correspond to stationary but unstable orbits, which makes the numerical work quite laborious. However, when these models are defined on Cayley trees, as in the case of the Ising model with competing interactions examined by Vannimenus (1981), it turns out that physically interesting solutions correspond to the attractors of the mapping. This simplifies the numerical work considerably, and detailed study of the whole phase diagram becomes feasible. Apart from the intrinsic interest attached to the study of models on trees, it is possible to argue that the results obtained on trees provide a useful guide to the more involved study of their counterparts on crystal lattices.

In this letter we analyse the chiral Potts model on a Cayley tree, or rather on a Bethe lattice (Baxter 1982), since we are concerned with the local properties of the innermost region. All the sites have coordination number z, except the outermost sites (first generation) and the innermost site (last generation) which have coordination numbers 1 and $z-1$, respectively. Associated with each site i there is a p-state spin S_i, which may assume the values $e_n = (\cos 2\pi n/p, \sin 2\pi n/p, 0)$, with $n = 0, 1, 2, \ldots, p-1$. The Hamiltonian contains two competing interactions; the usual exchange interactions which tend to align the spins and the Dzyaloshinsky–Moriya

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interactions which tend to rotate the spins, i.e.,

$$\mathcal{H} = -J_1 \sum_{(ij)} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - J_2 \sum_{(ij)} (\boldsymbol{S}_i \times \boldsymbol{S}_j) \cdot \hat{z},$$

where the summations are over nearest-neighbour pairs such that $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$ belong to the lth and $(l+1)$th generations, respectively. The Hamiltonian is customarily written in the form (Ostlund 1981)

$$\mathcal{H} = -J \sum_{(ij} \cos \frac{2\pi}{p}(n_i - n_j + \Delta),$$

where J and the chiral field Δ are related to J_1 and J_2 through $J_1 = J\cos(2\pi\Delta/p)$ and $J_2 = J\sin(2\pi\Delta/p)$.

Let $\rho_N^{(n)}$ be the probability of the innermost spin of an Nth generation tree being in the state n. One can relate $\rho_N^{(n)}$ to $\rho_{N-1}^{(n')}$ according to standard procedures (Baxter 1982), with the result

$$\rho_N^{(n)} = \frac{1}{\mathcal{N}_{N-1}} \left(\sum_{n'=0}^{p-1} \rho_{N-1}^{(n')} \exp \beta J \cos \frac{2\pi}{p}(n - n' - \Delta) \right)^{z-1},$$

where $\mathcal{N}_{n-1}$ is the normalisation factor such that $\Sigma_{n=0}^{p-1} \rho_N^{(n)} = 1$. This represents a $(p-1)$-dimensional mapping on the variables $\rho^{(n)}$. The interesting fact is that when the 'mean-field limit' $z \to \infty$ is taken, with $Jz = \bar{J} =$ constant, the mapping reduces to a two-dimensional one, irrespective of the number of states p. As a matter of fact, it is possible to express the mapping directly in terms of the complex magnetisation, $m = m_x + \mathrm{i}m_y$, in the form

$$m_N = \sum_{n=0}^{p-1} \rho_{N-1}^{(n)} e_n = \frac{1}{\mathcal{N}_{N-1}} \sum_{n=0}^{p-1} e_n \exp\left(\frac{1}{T} \exp(\mathrm{i}2\pi\Delta/p) m_{N-1} e_n^* + \mathrm{cc} \right) \quad (1)$$

where $e_0, e_1, \ldots, e_{p-1}$ are the p roots of unity, and $T = 2/\beta\bar{J}$ is the reduced temperature. In what follows we will report the main findings concerning the three-state model $(p = 3)$ in the mean-field limit $(z \to \infty)$.

The mapping (1) can best be visualised in the complex $m = m_x + \mathrm{i}m_y$ plane where the magnetisation of each generation is represented by a point inside the equilateral triangle with vertices e_0, e_1, and e_2. The equilibrium configuration for a given T and Δ may be found by the repeated iteration of the recursion relation (1), starting from an initial or surface magnetisation $m_1 \neq 0$. Numerically, we find that the magnetisation always approaches one of the following attractors: (a) the trivial fixed point (paramagnetic phase), (b) the non-trivial fixed points (ferromagnetic phase), (c) periodic cycles (commensurate phase) and (d) one-dimensional orbits (incommensurate phase). The (largest) Lyapunov number (Ott 1981) λ is negative for (a)-(c) and zero for (d) within the numerical accuracy. The possibility of a strange attractor with $\lambda > 0$ (Ott 1981), which would correspond to a chaotic phase (Bak 1981, Bak 1982, Jensen and Bak 1983), cannot be ruled out *a priori* but it was not found in the present model. The principal wavenumber q is obtained as $2\pi n/N$, where n is the number of turns of the magnetisation vector in N iterations, after discarding a sufficient number of initial iterations. Figure 1 represents the global phase diagram constructed in this way. Notice that only the interval $0 \leqslant \Delta \leqslant 0.5$ will be considered because of the symmetry relations $\Delta \to -\Delta$, $q \to -q$ and $\Delta \to 1 + \Delta$, $q \to q + 2\pi/3$, which follow from equation (1). In certain regions of the phase diagram more than one type of attractor exists, which can be interpreted as phase coexistence or first-order transition. Figure 2 displays the flow diagram corresponding to a point where ferromagnetic and modulated phases overlap.

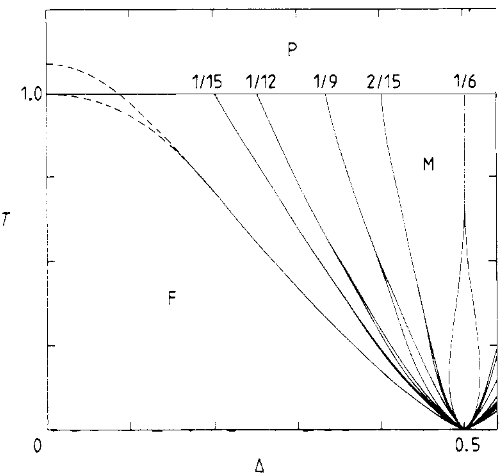

Figure 1. Phase diagram of the three-state chiral Potts model on a Cayley tree in the 'mean-field limit'. Paramagnetic (P), ferromagnetic (F) and modulated (M) phases correspond to the regions where the trivial fixed point, the non-trivial fixed points and the limit cycles, respectively, are the attractors. In the M region only a few commensurate phases are shown. Between the broken curves the F region overlaps the P region for $T > 1$ and the M region for $T < 1$.

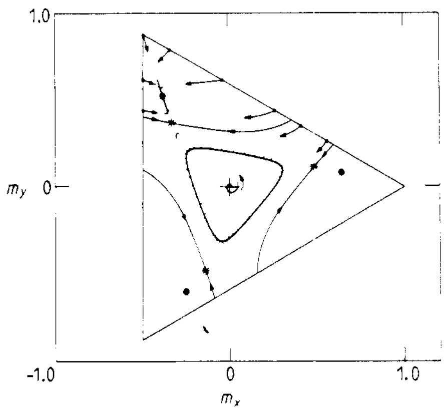

Figure 2. The attractors of the mapping (1) for $T = 0.95$ and $\Delta = 0.106$. Some flows resulting from different starting points are shown. Notice the coexistence of two types of attractors, the ferromagnetic fixed points (●) and the one-dimensional orbit. The three separatrices and unstable fixed points (*) are also shown.

The triangle is divided into four regions, three of them dominated by ferromagnetic fixed points, and a central region under the influence of a one-dimensional orbit.

One of the most intriguing questions related to systems with modulated ordering refers to the behaviour of commensurate and incommensurate phases as a function of temperature (Fisher and Huse 1982). Fortunately, in the present model, commensurate phases can be computed accurately to a high order and a convincing answer can be given. Figure 3 shows the wavenumber against chiral field curve for three different temperatures. Numerically we observe that commensurate phases become stable at every rotational value of $q/2\pi$. Moreover, all commensurate phases with wavenumber of the form $q = 2\pi m/3n$, for given n, have roughly the same width. The curve bears some resemblance to the Cantor's devil's staircase (Mandelbrot 1977) at low temperatures. The fractal dimensionality (Mandelbrot 1977) D_F associated with the set of points in between the level treads (commensurate phases) can be thought of as a measure of the relative importance of the incommensurate phases. D_F may be estimated (Mandelbrot 1977) by the slope of $\log(L(\varepsilon)/\varepsilon)$ as a function of $\log(1/\varepsilon)$, where ε is the scale which we choose to be the widths of the phases $q = 2\pi/3n$, and $L(\varepsilon) = 0.5 - S(\varepsilon)$, where $S(\varepsilon)$ is the total length of the commensurate phases with widths larger than ε. We observe that D_F should be interpreted as an 'average' fractal dimensionality, for the staircase lacks perfect self-similarity. In figure 4 we show the result for various temperatures. Since $L(\varepsilon) \sim \varepsilon^{1-D_F}$, $D_F < 1$ implies that incommensurate phases have zero measure and the devil's staircase is complete (Bak 1982). However, when D_F reaches unity their measure becomes non-zero and the staircase is incomplete (Bak 1982). Accordingly, as shown in inset of figure 4, the devil's staircase changes from complete to incomplete as the temperature is raised.

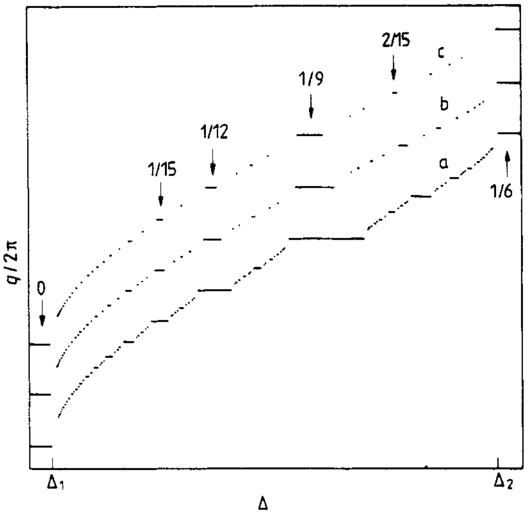

Figure 3. Devil's staircase for three different temperatures. The staircase are shifted along the q axis and the scale in the Δ axis is different for each temperature. Only commensurate phases with widths larger than about 0.000 03 are shown. (a) $T_1=0.15$ ($\Delta_1=0.412\,308$, $\Delta_2=0.481\,695$); (b) $T_2=0.25$ ($\Delta_1=0.367\,924$, $\Delta_2=0.483\,992$); (c) $T_3=0.30$ ($\Delta_1=0.347\,231$, $\Delta_2=0.486\,488$).

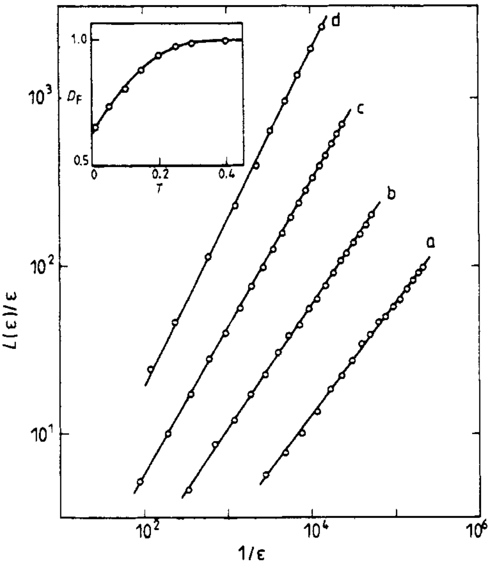

Figure 4. Log–log plot of $L(\varepsilon)/\varepsilon$ as a function of $1/\varepsilon$ for various temperatures. The slope of the straight line fitted to the points gives the fractal dimensionality D_F associated with the devil's staircase. The inset shows the temperature dependence of D_F. (a) $T=0.01$ (b) $T=0.05$ (c) $T=0.15$ (d) $T=0.40$.

Even though at intermediate temperatures numerical solution is the only viable alternative, at high and low temperatures analytic study becomes feasible. Thus, at high temperatures one can develop the RHS of equation (1) in power series and seek solution of the form $m_N=m_q\exp(\mathrm{i}qN)+m_{-2q}\exp(-2\mathrm{i}qN)+m_{4q}\exp(4\mathrm{i}qN)+m_{-5q}\exp(-5\mathrm{i}qN)+\ldots$. To leading order in $t=T_c-T=1-T$ we obtain

$$q=\tfrac{2}{3}\pi\Delta-\tfrac{1}{3}t\cot(\pi\Delta)+\mathrm{O}(t^2)$$

$$m_q=2(t/3)^{1/2}\exp(\mathrm{i}\phi)+\mathrm{O}(t^{3/2}),$$

$$m_{-2q}=\tfrac{1}{3}t\operatorname{cosec}(\pi\Delta)\exp[\mathrm{i}(-\pi\Delta+\tfrac{1}{2}\pi-2\phi)]+\mathrm{O}(t^2),$$

and in general $m_{\pm nq}\sim t^{n/2}$. The phase angle ϕ is arbitrary for incommensurate wavenumber q. For commensurate wavenumber of the form $q=2\pi m/3n$ the phase angle ϕ is determined by the $(3n-1)$th term in the expansion. For example, for $q=2\pi/6$ the 5th-order term implies that ϕ should vary from generation to generation as

$$\phi_N=\phi_{N-1}+\tfrac{2}{3}\pi\Delta-\tfrac{1}{3}\pi-\tfrac{1}{180}t^2\sin 6\phi_{N-1}+\mathrm{O}(t^3). \qquad (2)$$

For $\Delta>\Delta_c=\frac{1}{2}-t^2/120\pi+\mathrm{O}(t^3)$ the iteration of the recursion relation (2) leads always to a fixed point; this corresponds to the stability region of the commensurate phase $q=2\pi/6$. For $\Delta<\Delta_c$ a tangent bifurcation occurs (Eckmann 1981, Fisher and Huse 1982), and ϕ is trapped for very many iterations near the values $(2n+1)\pi/12$, and varies widely between these values, which correspond to 'discommensurations' or 'defects' (Bak 1982, Vannimenus 1981, Fisher and Huse 1982). As shown in the theory of intermittency (Manneville and Pomeau 1980) the number of iterations executed near these quasi-6-cycles is proportional to $|\Delta-\Delta_c|^{-1/2}$, so the wavenumber approaches $\pi/6$ as $\pi/6-q\sim|\Delta-\Delta_c|^{1/2}$. The stability of higher-order commensurate phases can be studied analogously. In general we observe that the width of a commensurate phase $q=2\pi m/3n$ should approach zero at $T=T_c=1$ as $t^{(3n-2)/2}$ in the form of sharp cusps. The exception is the ferromagnetic phase which has a flat parabolic boundary given asymptotically by $\Delta_c\simeq(3/2\pi)(\frac{2}{3}t)^{1/2}$.

At low temperatures the magnetisation is exponentially close to the sides of the equilateral triangle. In the ferromagnetic phase close to the multiphase point $\Delta=0.5$ the phase angle ϕ obeys asymptotically the recursion relation

$$\phi_N\simeq\frac{\sqrt{3}}{2}\exp\left[\frac{-4\pi\sqrt{3}}{3T}\left(\frac{1}{2}-\Delta\right)+\frac{2\sqrt{3}}{T}\phi_{N-1}\right].$$

The fixed point exists for $\Delta<\Delta_c\simeq\frac{1}{2}+(\sqrt{3}/4\pi)T\ln T$. For $\Delta>\Delta_c$ a tangent bifurcation occurs and the previous discussion applies. In particular the envelope of the commensurate wavenumbers near Δ_c should vary as $|\Delta-\Delta_c|^{1/2}$. Other phase boundaries can be analysed in similar fashion, and they all meet at the multiphase point with zero derivative.

The model we have studied here, although defined on a tree, shares many properties with models defined on crystal lattices (Bak 1982, Ostlund 1981), including the existence of complete and incomplete devil's staircase. However, some peculiarities are worth pointing out, such as the square root dependence of the wavenumber near commensurate phases, and the fact that the transition remains continuous down to $T=0$. A more thorough study of the p-state model on a z-coordinate tree is under way.

We would like to acknowledge Professor Per Bak for useful discussions in the early stages of this work, and Professor Herrmann for calling our attention to his related work on the same model. Fesser and Herrmann (1984), however, have analysed completely different aspects of the model.

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